

SIMULTANEOUS UNITARY EQUIVALENCE TO BI-CARLEMAN OPERATORS WITH ARBITRARILY SMOOTH KERNELS OF MERCER TYPE

IGOR M. NOVITSKIĬ

ABSTRACT. In this paper, we characterize the families of those bounded linear operators on a separable Hilbert space which are simultaneously unitarily equivalent to integral bi-Carleman operators on $L_2(\mathbb{R})$ having *arbitrarily* smooth kernels of Mercer type. The main result is a qualitative sharpening of an earlier result of [7].

1. INTRODUCTION. MAIN RESULT

Throughout, \mathcal{H} will denote a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$, $\mathfrak{K}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} , and \mathbb{C} , and \mathbb{N} , and \mathbb{Z} , the complex plane, the set of all positive integers, the set of all integers, respectively. For an operator A in $\mathfrak{K}(\mathcal{H})$, A^* will denote the Hilbert space adjoint of A in $\mathfrak{K}(\mathcal{H})$. Given an operator $T \in \mathfrak{K}(\mathcal{H})$, define an operator set

$$\mathcal{M}(T) = (T\mathfrak{K}(\mathcal{H}) \cup T^*\mathfrak{K}(\mathcal{H})) \cap (\mathfrak{K}(\mathcal{H})T^* \cup \mathfrak{K}(\mathcal{H})T),$$

where $S\mathfrak{K}(\mathcal{H})$, $\mathfrak{K}(\mathcal{H})S$ stand for the sets

$$\{SA \mid A \in \mathfrak{K}(\mathcal{H})\}, \quad \{AS \mid A \in \mathfrak{K}(\mathcal{H})\},$$

respectively.

Throughout, $C(X, B)$, where B is a Banach space (with norm $\|\cdot\|_B$), denote the Banach space (with the norm $\|f\|_{C(X, B)} = \sup_{x \in X} \|f(x)\|_B$) of continuous B -valued functions defined on a locally compact space X and *vanishing at infinity* (that is, given any $f \in C(X, B)$ and $\varepsilon > 0$, there exists a compact subset $X(\varepsilon, f) \subset X$ such that $\|f(x)\|_B < \varepsilon$ whenever $x \notin X(\varepsilon, f)$).

Let \mathbb{R} be the real line $(-\infty, +\infty)$ with the Lebesgue measure, and let $L_2 = L_2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) measurable

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complex-valued functions on \mathbb{R} equipped with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(s) \overline{g(s)} ds$$

and the norm $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$.

A linear operator $T : L_2 \rightarrow L_2$ is said to be *integral* if there exists a measurable function \mathbf{T} on the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, a *kernel*, such that, for every $f \in L_2$,

$$(Tf)(s) = \int_{\mathbb{R}} \mathbf{T}(s, t) f(t) dt$$

for almost every s in \mathbb{R} . A kernel \mathbf{T} on \mathbb{R}^2 is said to be *Carleman* if $\mathbf{T}(s, \cdot) \in L_2$ for almost every fixed s in \mathbb{R} . An integral operator with a kernel \mathbf{T} is called *Carleman* if \mathbf{T} is a Carleman kernel, and it is called *bi-Carleman* if both \mathbf{T} and \mathbf{T}^* ($\mathbf{T}^*(s, t) = \overline{\mathbf{T}(t, s)}$) are Carleman kernels. Every Carleman kernel, \mathbf{T} , induces a *Carleman function* \mathbf{t} from \mathbb{R} to L_2 by $\mathbf{t}(s) = \overline{\mathbf{T}(s, \cdot)}$ for all s in \mathbb{R} for which $\mathbf{T}(s, \cdot) \in L_2$.

We shall also recall a characterization of bi-Carleman representable operators. Its version for self-adjoint operators was first obtained by von Neumann [10] and was later extended by Korotkov to the general case (see [4, p. 100], [2, p. 103]). The assertion says that a necessary and sufficient condition that an operator $S \in \mathfrak{R}(\mathcal{H})$ be unitarily equivalent to a bi-Carleman operator is that there exist an orthonormal sequence $\{e_n\}$ such that

$$(1) \quad \|Se_n\|_{\mathcal{H}} \rightarrow 0, \quad \|S^*e_n\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

(or, equivalently, that 0 belong to the essential spectrum of $SS^* + S^*S$).

Definition 1. Given any non-negative integer m , we say that a function \mathbf{K} on \mathbb{R}^2 is a K^m -kernel (see [7], [6]) if

- (i) the function \mathbf{K} and all its partial derivatives on \mathbb{R}^2 up to order m are in $C(\mathbb{R}^2, \mathbb{C})$,
- (ii) the Carleman function \mathbf{k} , $\mathbf{k}(s) = \overline{\mathbf{K}(s, \cdot)}$, and all its (strong) derivatives on \mathbb{R} up to order m are in $C(\mathbb{R}, L_2)$,
- (iii) the conjugate transpose function \mathbf{K}^* , $\mathbf{K}^*(s, t) = \overline{\mathbf{K}(t, s)}$, satisfies Condition (ii), that is, the Carleman function \mathbf{k}^* , $\mathbf{k}^*(s) = \overline{\mathbf{K}^*(s, \cdot)}$, and all its (strong) derivatives on \mathbb{R} up to order m are in $C(\mathbb{R}, L_2)$.

In addition, we say that a function \mathbf{K} is a K^∞ -kernel (see [8], [9]) if it is a K^m -kernel for each non-negative integer m .

Definition 2. Let \mathbf{K} be a $K^m(K^\infty)$ -kernel and let T be the integral operator it induces. We say that the $K^m(K^\infty)$ -kernel \mathbf{K} is of *Mercer type* if every operator $A \in \mathcal{M}(T)$ is an integral operator having $K^m(K^\infty)$ -kernel.

The concept of Mercer type K^m -kernels for finite m was first introduced in our paper [7] where there is a motivation of the reason why this subclass of K^m -kernels deserves the qualification “of Mercer type”.

Given any non-negative integer m , the following result both gives a characterization of all bounded operators whose unitary orbits contain a bi-Carleman operator having K^m -kernel of Mercer type and describes families of those operators that can be simultaneously unitarily represented as bi-Carleman operators having K^m -kernels of Mercer type (cf. (1)).

Proposition ([7]). *If for an operator family $\{S_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathfrak{R}(\mathcal{H})$ there exists an orthonormal sequence $\{e_n\}$ such that*

$$\lim_{n \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \|S_\alpha^* e_n\|_{\mathcal{H}} = 0, \quad \lim_{n \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \|S_\alpha e_n\|_{\mathcal{H}} = 0,$$

then there exists a unitary operator $U_m : \mathcal{H} \rightarrow L_2$ such that all the operators $U_m S_\alpha U_m^{-1}$ ($\alpha \in \mathcal{A}$) and their linear combinations are bi-Carleman operators having K^m -kernels of Mercer type.

The construction of the unitary operator U_m given in the proof of Proposition depends on the preassigned order $m < \infty$ of smoothness (see [7]). The purpose of the present paper is to show that Proposition is true with K^∞ -kernels in the conclusion, that is, to prove the following qualitative sharpening of Proposition.

Theorem. *If for an operator family $\{S_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathfrak{R}(\mathcal{H})$ there exists an orthonormal sequence $\{e_n\}$ such that*

$$(2) \quad \lim_{n \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \|S_\alpha^* e_n\|_{\mathcal{H}} = 0, \quad \lim_{n \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \|S_\alpha e_n\|_{\mathcal{H}} = 0,$$

then there exists a unitary operator $U_\infty : \mathcal{H} \rightarrow L_2$ such that all the operators $U_\infty S_\alpha U_\infty^{-1}$ ($\alpha \in \mathcal{A}$) and their linear combinations are bi-Carleman operators having K^∞ -kernels of Mercer type.

2. PROOF OF THEOREM

The proof is broken up into three steps. The first step is to find suitable orthonormal bases $\{u_n\}$ in L_2 and $\{f_n\}$ in \mathcal{H} on which the construction of U_∞ will be based. The next step is to define a certain unitary operator that sends the basis $\{f_n\}$ onto the basis $\{u_n\}$. This operator is suggested as U_∞ in the theorem, and the rest of the proof is a straightforward verification that it is indeed as desired. Thus, the proof yields more than just existence of the unitary equivalence; it yields an explicit construction of the unitary operator. From the point of view of the applications to operator equations, the explicit computability of U_∞ is an important side issue.

Step 1. For the proof, it will be convenient to have the following notation: if an equivalence class $f \in L_2$ contains a function belonging to $C(\mathbb{R}, \mathbb{C})$, then we shall use $[f]$ to denote that function.

Let $\{S_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathfrak{R}(\mathcal{H})$ be a family satisfying (2) with the orthonormal sequence $\{e_n\}_{n=1}^\infty$. Take orthonormal bases $\{f_n\}$ for \mathcal{H} and $\{u_n\}$ for L_2 which satisfy the conditions:

- (a) the terms of the sequence $\{[u_n]^{(i)}\}$ of derivatives are in $C(\mathbb{R}, \mathbb{C})$, for each i (here and throughout, the letter i is reserved for integers in $[0, +\infty)$),
- (b) $\{u_n\} = \{g_k\}_{k=1}^\infty \cup \{h_k\}_{k=1}^\infty$, where $\{g_k\}_{k=1}^\infty \cap \{h_k\}_{k=1}^\infty = \emptyset$, and, for each i ,

$$(3) \quad \sum_k H_{k,i} < \infty \quad \text{with } H_{k,i} = \|[h_k]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} \quad (k \in \mathbb{N})$$

(the sum notation \sum_k will always be used instead of the more detailed

symbol $\sum_{k=1}^\infty$),

- (c) $\{f_n\} = \{x_k\}_{k=1}^\infty \cup \{y_k\}_{k=1}^\infty$ where $\{x_k\}_{k=1}^\infty \cap \{y_k\}_{k=1}^\infty = \emptyset$, $\{x_k\}_{k=1}^\infty \subset \{e_n\}_{n=1}^\infty$, and, for each i ,

$$(4) \quad \sum_k d_k (G_{k,i} + 1) < \infty$$

with $d_k = 2 \left(\sup_\alpha \|S_\alpha x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \sup_\alpha \|S_\alpha^* x_k\|_{\mathcal{H}}^{\frac{1}{4}} \right) \leq 1$, and $G_{k,i} = \|[g_k]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} \quad (k \in \mathbb{N})$.

The proof uses the bases just described to construct the desired unitary operator U_∞ .

Remark. Let $\{u_n\}$ be an orthonormal basis for L_2 such that, for each i ,

$$(5) \quad [u_n]^{(i)} \in C(\mathbb{R}, \mathbb{C}) \quad (n \in \mathbb{N}),$$

$$(6) \quad \|[u_n]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} \leq D_n A_i \quad (n \in \mathbb{N}),$$

$$(7) \quad \sum_k D_{n_k} < \infty,$$

where $\{D_n\}_{n=1}^\infty$, $\{A_i\}_{i=0}^\infty$ are sequences of positive numbers, and $\{n_k\}_{k=1}^\infty$ is a subsequence of \mathbb{N} such that $\mathbb{N} \setminus \{n_k\}_{k=1}^\infty$ is a countable set. Since $d(e_n) \rightarrow 0$ as $n \rightarrow \infty$, it follows that there exists a subset $\{x_k\}_{k=1}^\infty \subset \{e_n\}_{n=1}^\infty$ for which Condition (4) holds with $\{g_k\}_{k=1}^\infty = \{u_n\} \setminus \{u_{n_k}\}_{k=1}^\infty$. Moreover, the properties (6) and (7) imply Condition (3) for $h_k = u_{n_k}$ ($k \in \mathbb{N}$). Complete the set $\{x_k\}_{k=1}^\infty$ to an orthonormal basis, and let y_k ($k \in \mathbb{N}$) denote the new elements of that basis. Then the bases $\{f_n\} = \{x_k\}_{k=1}^\infty \cup \{y_k\}_{k=1}^\infty$ and $\{u_n\}$ satisfy Conditions (a)-(c).

A good example of the basis satisfying (5)-(7) is a basis generated by the Lemarié-Meyer wavelet

$$u(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi(\frac{1}{2}+s)} \text{sign } \xi b(|\xi|) d\xi \quad (s \in \mathbb{R}),$$

with the bell function b belonging to $C^\infty(\mathbb{R})$ (for construction of the Lemarié-Meyer wavelets we refer to [5], [1, § 4], [3, Example D, p. 62]). In this case, u belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$, and hence all the derivatives $[u]^{(i)}$ are

in $C(\mathbb{R}, \mathbb{C})$. The corresponding orthonormal basis for L_2 is given by

$$u_{jk}(s) = 2^{\frac{j}{2}} u(2^j s - k) \quad (j, k \in \mathbb{Z}).$$

Rearrange, in a completely arbitrary manner, the orthonormal set $\{u_{jk}\}_{j,k \in \mathbb{Z}}$ into a simple sequence, so that it becomes $\{u_n\}_{n \in \mathbb{N}}$. Since, in view of this rearrangement, to each $n \in \mathbb{N}$ there corresponds a unique pair of integers j_n, k_n , and conversely, we can write, for each i ,

$$\|[u_n]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} = \|[u_{j_n k_n}]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} \leq D_n A_i,$$

where

$$D_n = \begin{cases} 2^{j_n^2} & \text{if } j_n > 0, \\ \left(\frac{1}{\sqrt{2}}\right)^{|j_n|} & \text{if } j_n \leq 0, \end{cases} \quad A_i = 2^{(i+\frac{1}{2})^2} \|[u]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})}.$$

Whence it follows that if $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ is a subsequence such that $j_{n_k} \rightarrow -\infty$ as $k \rightarrow \infty$, then

$$\sum_k D_{n_k} < \infty.$$

Thus, the basis $\{u_n\}$ satisfies Conditions (5)-(7).

Step 2. In this step our intention is to construct a candidate for the desired unitary operator U_∞ in the theorem. Define such a unitary operator $U_\infty : \mathcal{H} \rightarrow L_2$ on the basis vectors by setting

$$(8) \quad U_\infty x_k = g_k, \quad U_\infty y_k = h_k \quad \text{for all } k \in \mathbb{N},$$

in the harmless assumption that $U_\infty f_n = u_n$ for all $n \in \mathbb{N}$.

Step 3. The verification that U_∞ in (8) has the desired properties is straightforward. Fix an arbitrary $\alpha \in \mathcal{A}$ and put $T = U_\infty S_\alpha U_\infty^{-1}$. Once this is done, the index α may be omitted for S_α .

Let E be the orthogonal projection onto the closed linear span of the vectors x_k ($k \in \mathbb{N}$). Split the operator S as follows:

$$(9) \quad S = (1 - E)S + ES, \quad S^* = (1 - E)S^* + ES^*.$$

The operators $J = SE$ and $\tilde{J} = S^*E$ are nuclear operators and, therefore, are Hilbert-Schmidt operators; these properties are almost immediate consequences of (4).

Write the Schmidt decompositions

$$J = \sum_n s_n \langle \cdot, p_n \rangle_{\mathcal{H}} q_n, \quad \tilde{J} = \sum_n \tilde{s}_n \langle \cdot, \tilde{p}_n \rangle_{\mathcal{H}} \tilde{q}_n,$$

where the s_n are the singular values of J (eigenvalues of $(JJ^*)^{\frac{1}{2}}$), $\{p_n\}, \{q_n\}$ are orthonormal sets (the p_n are eigenvectors for J^*J and q_n are eigenvectors for JJ^*). The explanation of the notation for \tilde{J} is similar.

Now introduce auxiliary operators B, \tilde{B} by

$$(10) \quad B = \sum_n s_n^{\frac{1}{4}} \langle \cdot, p_n \rangle_{\mathcal{H}} q_n, \quad \tilde{B} = \sum_n \tilde{s}_n^{\frac{1}{4}} \langle \cdot, \tilde{p}_n \rangle_{\mathcal{H}} \tilde{q}_n.$$

The Schwarz inequality yields

$$\begin{aligned}
 & \|B^*x_k\|_{\mathcal{H}} + \|Bx_k\|_{\mathcal{H}} + \|\tilde{B}^*x_k\|_{\mathcal{H}} + \|\tilde{B}x_k\|_{\mathcal{H}} \\
 &= \|(JJ^*)^{\frac{1}{8}}x_k\|_{\mathcal{H}} + \|(J^*J)^{\frac{1}{8}}x_k\|_{\mathcal{H}} \\
 (11) \quad &+ \|(\tilde{J}\tilde{J}^*)^{\frac{1}{8}}x_k\|_{\mathcal{H}} + \|(\tilde{J}^*\tilde{J})^{\frac{1}{8}}x_k\|_{\mathcal{H}} \\
 &\leq \|J^*x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|Jx_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|\tilde{J}^*x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|\tilde{J}x_k\|_{\mathcal{H}}^{\frac{1}{4}} \leq d_k.
 \end{aligned}$$

It follows that all the operators B, \tilde{B} are nuclear operators (see (4)) and hence

$$(12) \quad \sum_n s_n^{\frac{1}{2}} < \infty, \quad \sum_n \tilde{s}_n^{\frac{1}{2}} < \infty.$$

Define $Q = (1 - E)S^*, \tilde{Q} = (1 - E)S$. Then Condition (c) provides the representations

$$\begin{aligned}
 Qf &= \sum_k \langle Qf, y_k \rangle_{\mathcal{H}} y_k = \sum_k \langle f, Sy_k \rangle_{\mathcal{H}} y_k, \\
 (13) \quad \tilde{Q}f &= \sum_k \langle \tilde{Q}f, y_k \rangle_{\mathcal{H}} y_k = \sum_k \langle f, S^*y_k \rangle_{\mathcal{H}} y_k,
 \end{aligned}$$

for all f in \mathcal{H} .

Using the decompositions (9), which now look like $S = \tilde{Q} + \tilde{J}^*, S^* = Q + J^*$, we shall prove presently that T is an integral operator having K^∞ -kernel of Mercer type.

From (13) and (8), it follows that, for each $f \in L_2$,

$$\begin{aligned}
 Pf &= U_\infty Q U_\infty^{-1} f = \sum_k \langle f, Th_k \rangle_{\mathcal{H}} h_k, \\
 (14) \quad \tilde{P}f &= U_\infty \tilde{Q} U_\infty^{-1} f = \sum_k \langle f, T^*h_k \rangle_{\mathcal{H}} h_k.
 \end{aligned}$$

Represent the equivalence classes Th_k, T^*h_k ($k \in \mathbb{N}$) by the Fourier expansions

$$Th_k = \sum_n \langle y_k, S^*f_n \rangle_{\mathcal{H}} u_n, \quad T^*h_k = \sum_n \langle y_k, Sf_n \rangle_{\mathcal{H}} u_n,$$

where the series converge in the L_2 sense. But more than that can be said about convergence, namely that, for each fixed i , the series

$$(15) \quad \sum_n \langle y_k, S^*f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle y_k, Sf_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})$$

converge in the norm of $C(\mathbb{R}, \mathbb{C})$. Indeed, all the series are everywhere pointwise dominated by one series

$$\sum_n (\|S^*f_n\|_{\mathcal{H}} + \|Sf_n\|_{\mathcal{H}}) |[u_n]^{(i)}(s)|,$$

which is uniformly convergent on \mathbb{R} for the following reason: its subseries

$$\sum_k (\|Sx_k\|_{\mathcal{H}} + \|S^*x_k\|_{\mathcal{H}}) |[g_k]^{(i)}(s)|,$$

$$\sum_k (\|Sy_k\|_{\mathcal{H}} + \|S^*y_k\|_{\mathcal{H}}) |[h_k]^{(i)}(s)|$$

are uniformly convergent on \mathbb{R} because they in turn are dominated by the convergent series

$$(16) \quad \sum_k d_k G_{k,i}, \quad \sum_k 2\|S\| H_{k,i},$$

respectively (see (4), (3)).

It is now evident that the pointwise sums in (15) define functions that belong to $C(\mathbb{R}, \mathbb{C})$. Moreover, the above arguments prove that, for each fixed i , the derivative sequences $\{[Th_k]^{(i)}\}$, $\{[T^*h_k]^{(i)}\}$ are uniformly bounded in $C(\mathbb{R}, \mathbb{C})$ in the sense that there exists a positive constant C_i such that

$$\|[Th_k]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} < C_i, \quad \|[T^*h_k]^{(i)}\|_{C(\mathbb{R}, \mathbb{C})} < C_i,$$

for all k . Hence, by (3), it is possible to infer that, for all non-negative integers i, j , both

$$\sum_k [h_k]^{(i)}(s) \overline{[Th_k]^{(j)}(t)} \quad \text{and} \quad \sum_k [h_k]^{(i)}(s) \overline{[T^*h_k]^{(j)}(t)}$$

converge in the norm of $C(\mathbb{R}^2, \mathbb{C})$. This makes it obvious that both

$$(17) \quad P(s, t) = \sum_k [h_k](s) \overline{[Th_k](t)}$$

and

$$\widetilde{P}(s, t) = \sum_k [h_k](s) \overline{[T^*h_k](t)},$$

satisfy Condition (i) for each m .

Now we prove that the (Carleman) functions

$$(18) \quad p(s) = \overline{P(s, \cdot)} = \sum_k \overline{[h_k](s)} Th_k,$$

$$\widetilde{p}(s) = \overline{\widetilde{P}(s, \cdot)} = \sum_k \overline{[h_k](s)} T^*h_k$$

satisfy Condition (ii) for all m . Indeed, the series displayed converge absolutely in the $C(\mathbb{R}, L_2)$ sense, because those two series whose terms are $|[h_k](s)| \|Th_k\|$ and $|[h_k](s)| \|T^*h_k\|$ respectively are dominated by the second series in (16) for $i = 0$. For the remaining i , a similar reasoning implies the same conclusion for the series

$$\sum_k \overline{[h_k]^{(i)}(s)} Th_k, \quad \sum_k \overline{[h_k]^{(i)}(s)} T^*h_k.$$

The asserted property of both p and \widetilde{p} to satisfy (ii) for each m then follows from the termwise differentiation theorem. Now observe that, by (3) and

(18), the series in (14) (viewed, of course, as ones with terms belonging to $C(\mathbb{R}, \mathbb{C})$) converge (absolutely) in $C(\mathbb{R}, \mathbb{C})$ -norm to the functions

$$\begin{aligned} [Pf](s) &\equiv \langle f, \mathbf{p}(s) \rangle \equiv \int_{\mathbb{R}} \mathbf{P}(s, t) f(t) dt, \\ [\tilde{P}f](s) &\equiv \langle f, \tilde{\mathbf{p}}(s) \rangle \equiv \int_{\mathbb{R}} \tilde{\mathbf{P}}(s, t) f(t) dt, \end{aligned}$$

respectively. Thus, both P and \tilde{P} are Carleman operators with \mathbf{P} and $\tilde{\mathbf{P}}$ their kernels, respectively, satisfying Conditions (i), (ii) for each m .

Now consider the (integral) Hilbert–Schmidt operators $F = U_{\infty} J^* U_{\infty}^{-1}$ and $\tilde{F} = U_{\infty} \tilde{J}^* U_{\infty}^{-1}$. Prove that both F and \tilde{F} have kernels satisfying (i) for each m . Starting from the Schmidt decompositions for F and \tilde{F} , define their kernels by

$$\begin{aligned} \mathbf{F}(s, t) &= \sum_n s_n^{\frac{1}{2}} [U_{\infty} B^* q_n](s) \overline{[U_{\infty} B p_n](t)}, \\ \tilde{\mathbf{F}}(s, t) &= \sum_n \tilde{s}_n^{\frac{1}{2}} [U_{\infty} \tilde{B}^* \tilde{q}_n](s) \overline{[U_{\infty} \tilde{B} \tilde{p}_n](t)}, \end{aligned} \quad (19)$$

for all s, t in \mathbb{R} , in the tacit assumption that the square brackets are everywhere permissible (for the auxiliary operators B, \tilde{B} see (10)). In view of (12) the desired conclusion that the kernels so defined satisfy (i) for each m can be inferred as soon as it is known that for each fixed i the terms of the sequences

$$\left\{ [U_{\infty} B p_k]^{(i)} \right\}, \quad \left\{ [U_{\infty} B^* q_k]^{(i)} \right\}, \quad \left\{ [U_{\infty} \tilde{B} \tilde{p}_k]^{(i)} \right\}, \quad \left\{ [U_{\infty} \tilde{B}^* \tilde{q}_k]^{(i)} \right\}$$

make sense, are in $C(\mathbb{R}, \mathbb{C})$, and are uniformly bounded in $C(\mathbb{R}, \mathbb{C})$.

To see the validity of the properties indicated, observe that all the series

$$\begin{aligned} \sum_n \langle p_k, B^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle q_k, B f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \\ \sum_n \langle \tilde{p}_k, \tilde{B}^* f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle \tilde{q}_k, \tilde{B} f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N}) \end{aligned}$$

(which in the case where $i = 0$ are just the Fourier expansions for $U_{\infty} B p_k$, $U_{\infty} B^* q_k$, $U_{\infty} \tilde{B} \tilde{p}_k$, $U_{\infty} \tilde{B}^* \tilde{q}_k$) are dominated by one series

$$\sum_n c(f_n) \left| [u_n]^{(i)}(s) \right|,$$

where $c(g) = \|B^* g\|_{\mathcal{H}} + \|B g\|_{\mathcal{H}} + \|\tilde{B}^* g\|_{\mathcal{H}} + \|\tilde{B} g\|_{\mathcal{H}}$ whenever $g \in \mathcal{H}$. The last series is uniformly convergent, because it consists of the two dominated and uniformly convergent subseries

$$\sum_n c(x_k) \left| [g_k]^{(i)}(s) \right|, \quad \sum_n c(y_k) \left| [h_k]^{(i)}(s) \right|;$$

the corresponding dominant series are

$$\sum_k d_k G_{k,i}, \quad \sum_k 2 \left(\|B\| + \|\tilde{B}\| \right) H_{k,i}$$

(see (11), (4), (3)).

In view of (12) and the uniform boundedness in $C(\mathbb{R}, \mathbb{C})$ of the sequences $\{[U_\infty B^* q_n]^{(i)}\}, \{[U_\infty \tilde{B}^* \tilde{q}_n]^{(i)}\}$ for each fixed i , the series

$$\sum_n s_n^{\frac{1}{2}} \overline{[U_\infty B^* q_n]^{(i)}}(s) U_\infty B p_n, \quad \sum_n \tilde{s}_n^{\frac{1}{2}} \overline{[U_\infty \tilde{B}^* \tilde{q}_n]^{(i)}}(s) U_\infty \tilde{B} \tilde{p}_n$$

are absolutely convergent in the $C(\mathbb{R}, L_2)$ sense, and hence their sums belong to $C(\mathbb{R}, L_2)$. Observe by (19) that two of them, namely those for $i = 0$, represent the Carleman functions $\mathbf{f}(s) = \overline{\mathbf{F}(s, \cdot)}$, $\tilde{\mathbf{f}}(s) = \overline{\tilde{\mathbf{F}}(s, \cdot)}$. Thus, both Carleman functions \mathbf{f} and $\tilde{\mathbf{f}}$ satisfy Condition (ii) for every m .

In accordance with (9), the operator T , which is the transform by U_∞ of S , has the decompositions $T = \tilde{P} + \tilde{F}$, $T^* = P + F$ where all the terms are the Carleman operators already described. So both T and T^* are Carleman operators, and their kernels \mathbf{K} and $\tilde{\mathbf{K}}$, which are defined by

$$(20) \quad \mathbf{K}(s, t) = \tilde{\mathbf{P}}(s, t) + \tilde{\mathbf{F}}(s, t), \quad \tilde{\mathbf{K}}(s, t) = \mathbf{P}(s, t) + \mathbf{F}(s, t),$$

for all $s, t \in \mathbb{R}$, inherit the two properties (i), (ii) from their terms, for each m . Since (cf. [2, p. 37]) it is possible to write $\mathbf{K}(s, t) = \overline{\tilde{\mathbf{K}}(t, s)}$ and $\mathbf{K}(\cdot, t) = \overline{\tilde{\mathbf{K}}(t, \cdot)}$ for all $s, t \in \mathbb{R}$, the kernel \mathbf{K} satisfies Conditions (i), (ii), (iii) for each m , so that it is a K^∞ -kernel.

As the preceding proof shows, the major condition that an operator $A \in \mathfrak{R}(\mathcal{H})$ must satisfy in order that $U_\infty A U_\infty^{-1}$ be an integral operator having K^∞ -kernel is that, for each k ,

$$2 \left(\|A x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|A^* x_k\|_{\mathcal{H}}^{\frac{1}{4}} \right) \leq d_k.$$

If $A \in \mathcal{M}(S)$ then there exist operators $V, W \in \mathfrak{R}(\mathcal{H})$ such that at least one of the relations $A = SV = WS$, $A = S^*V = WS^*$, $A = SV = WS^*$, $A = VS = S^*W$ holds. In any event, whatever its decomposition may be, the operator A satisfies the inequalities

$$\left(\|A x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|A^* x_k\|_{\mathcal{H}}^{\frac{1}{4}} \right) \leq 2c \left(\|A x_k\|_{\mathcal{H}}^{\frac{1}{4}} + \|A^* x_k\|_{\mathcal{H}}^{\frac{1}{4}} \right) \leq c d_k,$$

where $c^4 = \max \{\|V\|, \|W\|\}$. This implies, by the above remark, that U_∞ automatically carries every $A \in \mathcal{M}(S)$ onto an integral operator $U_\infty A U_\infty^{-1}$ having K^∞ -kernel so that \mathbf{K} in (20) is a K^∞ -kernel of Mercer type.

The fact that those K^∞ -kernels which induce finite linear combinations of $U_\infty S_\alpha U_\infty^{-1}$ are of Mercer type remains to be proved. The result can be inferred from the result for S , which has just been obtained. Indeed, consider any finite linear combination $G = \sum z_\alpha S_\alpha$ with $\sum |z_\alpha| \leq 1$. It is seen easily that, for each n ,

$$\left\| \sum z_\alpha S_\alpha e_n \right\|_{\mathcal{H}} \leq \sup_\alpha \|S_\alpha e_n\|_{\mathcal{H}}, \quad \left\| \sum \bar{z}_\alpha S_\alpha^* e_n \right\|_{\mathcal{H}} \leq \sup_\alpha \|S_\alpha^* e_n\|_{\mathcal{H}}.$$

There is, therefore, no barrier to assuming that G was, from the start, in $\{S_\alpha \mid \alpha \in \mathcal{A}\}$ and even equal to S . The proof of Theorem is complete.

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INSTITUTE FOR APPLIED MATHEMATICS, RUSSIAN ACADEMY OF SCIENCES, 92,
ZAPARINA STREET, Khabarovsk 680 000, RUSSIA
E-mail address: novim@iam.khv.ru